Resonant interactions between two trains of gravity waves

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In a previous paper, Phillips (1960) showed that two or three trains of gravity waves may interact so as to produce a fourth (tertiary) wave whose wave-number is different from any of the three primary wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, and whose amplitude grows in time. Such resonant interactions may produce an appreciable modification of the spectrum of ocean waves within a few hours. In this paper, by a slightly different method, the interaction is calculated in detail for the simplest possible case: when two of the three primary wave-numbers are equal ($\mathbf{k}_3 = \mathbf{k}_1$).

It is found that, when $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel or antiparallel, the interaction vanishes unless $\mathbf{k}_1 = \mathbf{k}_2$. Generally, if θ denotes the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$, the rate of growth of the tertiary wave with time is a maximum when $\theta \doteqdot 17°$; the rate of growth with horizontal distance is a maximum when $\theta \doteqdot 24°$. The calculations show that it should be possible to detect the tertiary wave in the laboratory.

1. Introduction

In the first-order theory of gravity waves of small amplitude, two or more simple sine-waves which each individually satisfy the condition of constant pressure at the free surface together satisfy the same condition, so that the wave trains are propagated independently and without mutual interaction. If now squares and products of the velocities are taken into account the waves are found to interact. To the second order, the interaction produces only a small modification to the motion, which remains bounded in time. However, Phillips (1960) has discovered that in the third approximation it is possible for a transfer of energy to take place from three primary waves (of wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$) to a fourth wave (of wave-number $\mathbf{k}_4$) in such a way that the amplitude of the fourth wave increases linearly with time. Thus, although the fourth-wave amplitude at first is very small (being of the third order) it may in time grow so as to be comparable with the three primary waves. The condition for this is that the wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$ and frequencies σ_1, σ_2, σ_3, σ_4 each satisfy the relation for a free wave:

$$\sigma_i^2 = g\,|\mathbf{k}_i| \quad (i = 1, 2, 3, 4), \tag{1.1}$$

and that

$$\sigma_1 \pm \sigma_2 \pm \sigma_3 \pm \sigma_4 = 0, \quad k_1 \pm k_2 \pm k_3 \pm k_4 = 0, \tag{1.2}$$

with the same combination of signs in each case.

If such interactions do occur in the oceans they may produce a considerable modification in the ocean wave spectrum, in a matter of a few hours. Indeed Hasselmann (1961) has computed the rate of transfer of energy within a continuous wave spectrum due to this kind of mechanism.

The phenomenon is so intriguing, and its possible effects of such consequence, that an attempt to verify it in the laboratory seems desirable. In Phillips's original paper only the order of magnitude of the interaction was estimated, and a calculation of the coupling factor was not given. The purpose of the present paper is to carry the calculations to the point where a numerical estimate can be made of the amplitude of the tertiary wave in the simplest possible case, namely when two of the wave-numbers $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ are equal. The computation shows that the tertiary wave might well be observed in some experimental wave basins at present available.

It is also thought that the simplified method of calculation, which differs in certain respects from that of Phillips or Hasselmann, may be of interest in itself.

2. General equations

The presence of a small vorticity (Longuet-Higgins 1953) will not affect the results to the degree of approximation considered, and so it is permissible to assume the existence of a potential ϕ for the velocity $\mathbf{u}$: thus

$$\mathbf{u} = \nabla\phi, \quad \nabla^2\phi = 0, \tag{2.1}$$

in incompressible flow. Let z be the vertical co-ordinate; then, at the free surface $z = \zeta$, the pressure being constant, we have from Bernoulli's equation

$$g\zeta + \frac{\partial\phi}{\partial t} + \tfrac{1}{2}\mathbf{u}^2 = 0 \quad (z = \zeta), \tag{2.2}$$

the arbitrary function of t being absorbed into ϕ. The condition that $(z-\zeta)$ vanish following a particle gives

$$\frac{\partial\zeta}{\partial t} - \frac{\partial\phi}{\partial z} + \left(\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}\right) = 0 \quad (z = \zeta). \tag{2.3}$$

By operating on equation (2.2) by D/Dt (differentiation following the motion) and then subtracting g times equation (2.3), we find that

$$\left(\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z}\right) + \frac{\partial}{\partial t}(\mathbf{u}^2) + \mathbf{u}\,.\,\nabla(\tfrac{1}{2}\mathbf{u}^2) = 0 \quad (z = \zeta). \tag{2.4}$$

Now let the left-hand sides of the above three equations be expanded in Taylor series about $z = 0$ to give

$$g\zeta + \left[\frac{\partial\phi}{\partial t} + \zeta\frac{\partial^2\phi}{\partial z\,\partial t} + \tfrac{1}{2}\zeta^2\frac{\partial^3\phi}{\partial z^2\,\partial t} + \dots\right] + \left[\tfrac{1}{2}\mathbf{u}^2 + \zeta\frac{\partial}{\partial z}(\tfrac{1}{2}\mathbf{u}^2) + \dots\right] = 0 \quad (z = 0), \tag{2.2a}$$

$$\frac{\partial\zeta}{\partial t} - \left[\frac{\partial\phi}{\partial z} + \zeta\frac{\partial^2\phi}{\partial z^2} + \tfrac{1}{2}\zeta^2\frac{\partial^3\phi}{\partial z^3} + \dots\right] + \left[\left(\frac{\partial\phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\zeta}{\partial y}\right) + \zeta\left(\frac{\partial^2\phi}{\partial x\,\partial z}\frac{\partial\zeta}{\partial x} + \frac{\partial^2\phi}{\partial y\,\partial z}\frac{\partial\zeta}{\partial y}\right) + \dots\right] = 0 \quad (z = 0), \tag{2.3a}$$

$$\left[\left(\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}\right)+\zeta\frac{\partial}{\partial z}\left(\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}\right)+\tfrac{1}{2}\zeta^2\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\phi}{\partial z}\right)+\dots\right]$$
$$+\left[\frac{\partial}{\partial t}(\mathbf{u}^2)+\zeta\frac{\partial^2}{\partial z\,\partial t}(\mathbf{u}^2)+\dots\right]+[\mathbf{u}\,.\nabla(\tfrac{1}{2}\mathbf{u}^2)+\dots]=0 \quad (z=0). \quad (2.4a)$$

In these equations we may substitute the formal expressions

$$\left.\begin{aligned}\phi &= (\alpha\phi_{10}+\beta\phi_{01})+(\alpha^2\phi_{20}+\alpha\beta\phi_{11}+\beta^2\phi_{02})+\dots,\\ \mathbf{u} &= (\alpha\mathbf{u}_{10}+\beta\mathbf{u}_{01})+(\alpha^2\mathbf{u}_{20}+\alpha\beta\mathbf{u}_{11}+\beta^2\mathbf{u}_{02})+\dots,\\ \zeta &= (\alpha\zeta_{10}+\beta\zeta_{01})+(\alpha^2\zeta_{20}+\alpha\beta\zeta_{11}+\beta^2\zeta_{02})+\dots,\end{aligned}\right\} \quad (2.5)$$

where $\alpha\phi_{10}$ and $\beta\phi_{01}$ are to represent the two intersecting wave trains, in the first approximation, α and β being independent small quantities proportional to the surface slopes. The remaining terms of the series represent wave interactions, and may be found by equating coefficients of $\alpha^i\beta^j$ in the various equations. Thus, from (2.1), we obtain

$$\mathbf{u}_{ij}=\nabla\phi_{ij}, \quad \nabla^2\phi_{ij}=0 \quad (i,j=1,2,\dots). \quad (2.6)$$

In (2.2a), (2.3a) and (2.4a) the terms in α give

$$g\zeta_{10}+\frac{\partial\phi_{10}}{\partial t}=0, \quad \frac{\partial\zeta_{10}}{\partial t}-\frac{\partial\phi_{10}}{\partial z}=0, \quad \frac{\partial^2\phi_{10}}{\partial t^2}+g\frac{\partial\phi_{10}}{\partial z}=0, \quad (2.7)$$

and the terms in β give similar equations for ϕ_{01}. In equation (2.4a) the terms in α^2 and $\alpha\beta$ give respectively

$$\left.\begin{aligned}\left(\frac{\partial^2\phi_{20}}{\partial t^2}+g\frac{\partial\phi_{20}}{\partial z}\right)+\zeta_{10}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{10}}{\partial t^2}+g\frac{\partial\phi_{10}}{\partial z}\right)+\frac{\partial}{\partial t}(\mathbf{u}_{10}^2)=0,\\ \left(\frac{\partial^2\phi_{11}}{\partial t^2}+g\frac{\partial\phi_{11}}{\partial z}\right)+\zeta_{10}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{01}}{\partial t^2}+g\frac{\partial\phi_{01}}{\partial z}\right)+\zeta_{01}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{10}}{\partial t^2}+g\frac{\partial\phi_{10}}{\partial z}\right)+\frac{\partial}{\partial t}(2\mathbf{u}_{10}.\mathbf{u}_{01})=0.\end{aligned}\right\} \quad (2.8)$$

Since ϕ_{10}, for example, satisfies Laplace's equation (2.6) we have

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\phi_{10}}{\partial t^2}+g\frac{\partial\phi_{10}}{\partial z}\right)=-\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2\phi_{10}}{\partial t^2}+g\frac{\partial\phi_{10}}{\partial z}\right)=0, \quad (2.9)$$

by the third of equation (2.7). It follows that in the equations for ϕ_{21} such terms can be omitted. Thus in (2.4a) the coefficient of $\alpha^2\beta$ gives

$$\left(\frac{\partial^2\phi_{21}}{\partial t^2}+g\frac{\partial\phi_{21}}{\partial z}\right)+\zeta_{20}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{01}}{\partial t^2}+g\frac{\partial\phi_{01}}{\partial z}\right)+\zeta_{01}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{20}}{\partial t^2}+g\frac{\partial\phi_{20}}{\partial z}\right)$$
$$+\zeta_{11}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{10}}{\partial t^2}+g\frac{\partial\phi_{10}}{\partial z}\right)+\zeta_{10}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{11}}{\partial t^2}+g\frac{\partial\phi_{11}}{\partial z}\right)$$
$$+\frac{\partial}{\partial t}(2\mathbf{u}_{20}.\mathbf{u}_{01}+2\mathbf{u}_{11}.\mathbf{u}_{10})+\zeta_{10}\frac{\partial^2}{\partial z\,\partial t}(2\mathbf{u}_{10}.\mathbf{u}_{01})+\zeta_{10}\frac{\partial^2}{\partial z\,\partial t}(\mathbf{u}_{10}^2)$$
$$+\mathbf{u}_{10}.\nabla(\mathbf{u}_{10}.\mathbf{u}_{01})+\mathbf{u}_{01}.\nabla(\tfrac{1}{2}\mathbf{u}_{10}^2)=0. \quad (2.10)$$

Equations (2.7), (2.8) and (2.10) are all to be satisfied when $z=0$.

3. Deep water

We shall now assume that the water is deep for the first- and second-order waves, i.e. that $e^{-k_1 h}, e^{-k_2 h}, e^{-k' h} \ll 1$, where $k_1 = |\mathbf{k}_1|$, $k_2 = |\mathbf{k}_2|$ and $k' = |\mathbf{k}_1 - \mathbf{k}_2|$. As yet, no assumption is made about the third-order waves.

The equations (2.7) for the first approximation ϕ_{10} are satisfied by the well known solution

$$\zeta_{10} = a_1 \cos \psi_1, \quad \phi_{10} = a_1 \sigma_1 k_1^{-1} e^{k_1 z} \sin \psi_1, \tag{3.1}$$

where for convenience we write

$$\psi_1 = \mathbf{k}_1 . \mathbf{x} - \sigma_1 t, \tag{3.2}$$

provided that

$$\sigma_1^2 = g k_1. \tag{3.3}$$

We have then

$$\left.\begin{aligned} \frac{\partial}{\partial z}\left(\frac{\partial^2 \phi_{10}}{\partial t^2} + g\frac{\partial \phi_{10}}{\partial z}\right) &= -k_1\left(\frac{\partial^2 \phi_{10}}{\partial z^2} + g\frac{\partial \phi_{10}}{\partial z}\right) = 0, \\ \mathbf{u}_{10}^2 &= a_1^2 \sigma_1^2 e^{2k_1 z}, \end{aligned}\right\} \tag{3.4}$$

so that $\mathbf{u}_{10}^2$ is independent of x, y and t.

The first of equations (2.8) is now satisfied identically by

$$\phi_{20} = \phi_{20}(t), \tag{3.5}$$

corresponding to the fact that the second-order velocities vanish for a single irrotational wave in deep water:

$$\mathbf{u}_{20} = \nabla \phi_{20} = 0. \tag{3.6}$$

(The second-order surface elevation ζ_{20} does not vanish, however.)

The velocity potential ϕ_{01} of the second wave is defined by equations similar to (3.1), (3.2) and satisfies relations similar to (3.3), (3.4). Moreover we find that

$$\mathbf{u}_{10} . \mathbf{u}_{01} = a_1 a_2 \sigma_1 \sigma_2 e^{(k_1 + k_2) z} [\cos^2 \tfrac{1}{2}\theta \cos(\psi_1 - \psi_2) - \sin^2 \tfrac{1}{2}\theta \cos(\psi_1 + \psi_2)], \tag{3.7}$$

where θ is the angle between the wave-numbers $\mathbf{k}_1$ and $\mathbf{k}_2$. Thus, from the second of equations (2.8), we have

$$\begin{aligned} -\left(\frac{\partial^2 \phi_{11}}{\partial t^2} + g\frac{\partial \phi_{11}}{\partial z}\right) = 2a_1 a_2 \sigma_1 \sigma_2 [(\sigma_1 - \sigma_2) \cos^2 \tfrac{1}{2}\theta \sin(\psi_1 - \psi_2) \\ - (\sigma_1 + \sigma_2) \sin^2 \tfrac{1}{2}\theta \sin(\psi_1 + \psi_2)], \end{aligned} \tag{3.8}$$

which is satisfied (together with Laplace's equation) by

$$\phi_{11} = A\, e^{|\mathbf{k}_1 - \mathbf{k}_2| z} \sin(\psi_1 - \psi_2) - B\, e^{|\mathbf{k}_1 + \mathbf{k}_2| z} \sin(\psi_1 + \psi_2), \tag{3.9}$$

where

$$\left.\begin{aligned} A &= \frac{2a_1 a_2 \sigma_1 \sigma_2 (\sigma_1 - \sigma_2) \cos^2 \frac{1}{2}\theta}{(\sigma_1 - \sigma_2)^2 - g|\mathbf{k}_1 - \mathbf{k}_2|}, \\ B &= \frac{2a_1 a_2 \sigma_1 \sigma_2 (\sigma_1 + \sigma_2) \sin^2 \frac{1}{2}\theta}{(\sigma_1 + \sigma_2)^2 - g|\mathbf{k}_1 + \mathbf{k}_2|}. \end{aligned}\right\} \tag{3.10}$$

The equation (2.10) for ϕ_{21} is now seen to be very greatly simplified. In fact, omitting those terms which are identically zero, we have

$$-\left(\frac{\partial^2\phi_{21}}{\partial t^2}+g\frac{\partial\phi_{21}}{\partial z}\right) = \zeta_{10}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{11}}{\partial t^2}+g\frac{\partial\phi_{11}}{\partial z}\right)+\frac{\partial}{\partial t}(2\mathbf{u}_{11}\cdot\mathbf{u}_{10})+\zeta_{10}\frac{\partial^2}{\partial z\partial t}(2\mathbf{u}_{10}\cdot\mathbf{u}_{01}) + \mathbf{u}_{10}\cdot\nabla(\mathbf{u}_{10}\cdot\mathbf{u}_{01})+\mathbf{u}_{01}\cdot\nabla(\tfrac{1}{2}\mathbf{u}_{10}^2), \quad (3.11)$$

to be satisfied at $z = 0$.

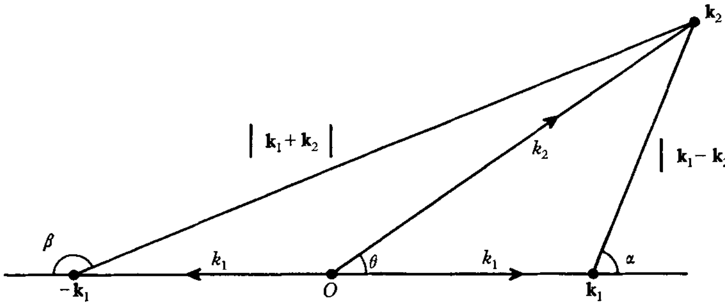

FIGURE 1. Definition diagram for α, β and θ.

The terms on the right-hand side of (3.11) when evaluated at $z = 0$ are as follows:

$$\zeta_{10}\frac{\partial}{\partial z}\left(\frac{\partial^2\phi_{11}}{\partial t^2}+g\frac{\partial\phi_{11}}{\partial z}\right) = -2a_1^2a_2\sigma_1\sigma_2[(\sigma_1-\sigma_2)\,|\mathbf{k}_1-\mathbf{k}_2|\cos^2\tfrac{1}{2}\theta\sin(\psi_1-\psi_2)\cos\psi_1 - (\sigma_1+\sigma_2)\,|\mathbf{k}_1+\mathbf{k}_2|\sin^2\tfrac{1}{2}\theta\sin(\psi_1+\psi_2)\cos\psi_1], \quad (3.12)$$

$$\frac{\partial}{\partial t}(2\mathbf{u}_{11}\cdot\mathbf{u}_{10}) = 2Aa_1\sigma_1\,|\mathbf{k}_1-\mathbf{k}_2|\,[\sigma_2\sin^2\tfrac{1}{2}\alpha\sin\psi_2-(2\sigma_1-\sigma_2)\cos^2\tfrac{1}{2}\alpha\sin(2\psi_1-\psi_2)] - 2Ba_1\sigma_1\,|\mathbf{k}_1+\mathbf{k}_2|\,[\sigma_2\sin^2\tfrac{1}{2}\beta\sin\psi_2-(2\sigma_1+\sigma_2)\cos^2\tfrac{1}{2}\beta\sin(2\psi_1+\psi_2)], \quad (3.13)$$

$$\zeta_{10}\frac{\partial^2}{\partial z\partial t}(2\mathbf{u}_{10}\cdot\mathbf{u}_{01}) = 2a_1^2a_2\sigma_1\sigma_2(k_1+k_2)[(\sigma_1-\sigma_2)\cos^2\tfrac{1}{2}\theta\sin(\psi_1-\psi_2)\cos\psi_1 - (\sigma_1+\sigma_2)\sin^2\tfrac{1}{2}\theta\sin(\psi_1+\psi_2)\cos\psi_1], \quad (3.14)$$

$$\mathbf{u}_{10}\cdot\nabla(\mathbf{u}_{10}\cdot\mathbf{u}_{01}) = a_1^2a_2\sigma_1^2\sigma_2[(k_1+k_2\cos^2\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta)\sin\psi_2 + k_2\cos^4\tfrac{1}{2}\theta\sin(2\psi_1-\psi_2)-k_2\sin^4\tfrac{1}{2}\theta\sin(2\psi_1+\psi_2)], \quad (3.15)$$

$$\mathbf{u}_{01}\cdot\nabla(\tfrac{1}{2}\mathbf{u}_{10}^2) = a_1^2a_2\sigma_1^2\sigma_2k_1\sin\psi_2, \quad (3.16)$$

where α and β are the angles between $(\mathbf{k}_2-\mathbf{k}_1)$ and $\mathbf{k}_1$, and between $(\mathbf{k}_1+\mathbf{k}_2)$ and $-\mathbf{k}_1$, respectively (see figure 1).

4. The resonant interaction

The right-hand side of (3.11) may be expressed as the sum of terms proportional to $\sin(\psi_1\pm\psi_2\pm\psi_1)$. In the present calculation we are interested only in the terms proportional to $\sin(2\psi_1-\psi_2)$. Omitting the others, we have

$$-\left(\frac{\partial^2\phi_{21}}{\partial t^2}+g\frac{\partial\phi_{21}}{\partial z}\right) = K\sin(2\psi_1-\psi_2), \quad (4.1)$$

where $$K = a_1^2 a_2 \sigma_1 \sigma_2 \cos^2 \tfrac{1}{2}\theta \left[(\sigma_1 - \sigma_2) \{(k_1 + k_2) - |\mathbf{k}_1 - \mathbf{k}_2|\} + \sigma_1 k_2 \cos^2 \tfrac{1}{2}\theta - \frac{4\sigma_1(\sigma_1 - \sigma_2)(2\sigma_1 - \sigma_2)|\mathbf{k}_1 - \mathbf{k}_2| \cos^2 \tfrac{1}{2}\alpha}{(\sigma_1 - \sigma_2)^2 - g|\mathbf{k}_1 - \mathbf{k}_2|} \right]. \tag{4.2}$$

If now we have $$(2\sigma_1 - \sigma_2)^2 = g|2\mathbf{k}_1 - \mathbf{k}_2|, \tag{4.3}$$

then a solution to equation (4.1) and of Laplace's equation is

$$-\phi_{21} = \frac{Kt}{2(2\sigma_1 - \sigma_2)} e^{|2\mathbf{k}_1 - \mathbf{k}_2|z} \cos(2\psi_1 - \psi_2), \tag{4.4}$$

provided that $e^{-|2\mathbf{k}_1 - \mathbf{k}_2|h} \ll 1$, i.e. the wave represented by ϕ_{21} is effectively in deep water.

Now (4.4) represents a wave whose amplitude grows in time. But from (2.2*a*), on equating coefficients of $\alpha^2\beta$ we have

$$g\zeta_{21} + \frac{\partial \phi_{21}}{\partial t} = f(\phi_{10}, \phi_{01}, \phi_{11}, \phi_{21}), \tag{4.5}$$

which is bounded in time. Hence

$$\zeta_{21} \simeq -\frac{1}{g}\frac{\partial \phi_{21}}{\partial t} \simeq \frac{Kt}{2g} \sin(2\psi_1 - \psi_2). \tag{4.6}$$

Thus the amplitude of this wave is simply $Kt/2g$.

5. The period equation

Equation (4.3), together with the relations

$$\sigma_1^2 = g|\mathbf{k}_1|, \quad \sigma_2^2 = g|\mathbf{k}_2|, \tag{5.1}$$

are together a special case of equations (1.1). We shall now investigate the relationship between $\mathbf{k}_1$ and $\mathbf{k}_2$ which these imply.

It is convenient to write

$$\mathbf{k}_2 - \mathbf{k}_1 = \mathbf{k}', \quad \sigma_2 - \sigma_1 = \sigma', \tag{5.2}$$

so that (5.1) and (4.3) become

$$\sigma_1^2 = g|\mathbf{k}_1|, \quad (\sigma_1 + \sigma')^2 = g|\mathbf{k}_1 + \mathbf{k}'|, \quad (\sigma_1 - \sigma')^2 = g|\mathbf{k}_1 - \mathbf{k}'|. \tag{5.3}$$

Since the angle between $\mathbf{k}_1$ and $\mathbf{k}'$ is α (see figure 1), we have

$$\left.\begin{aligned} (\sigma_1 + \sigma')^4 &= g^2|\mathbf{k}_1 + \mathbf{k}'|^2 = g^2(k_1^2 + k'^2 + 2k_1 k' \cos\alpha), \\ (\sigma_1 - \sigma')^4 &= g^2|\mathbf{k}_1 - \mathbf{k}'|^2 = g^2(k_1^2 + k'^2 - 2k_1 k' \cos\alpha), \end{aligned}\right\} \tag{5.4}$$

and so $$\left.\begin{aligned} \sigma_1^4 + 6\sigma_1^2\sigma'^2 + \sigma'^4 &= g^2(k_1^2 + k'^2), \\ 4\sigma_1^3\sigma' + 4\sigma_1\sigma'^3 &= g^2 2k_1 k' \cos\alpha. \end{aligned}\right\} \tag{5.5}$$

On substituting $\sigma_1^2 = gk_1$ in the last two equations, we find

$$gk' = \sigma'(6\sigma_1^2 + \sigma'^2)^{\frac{1}{2}}, \quad \cos\alpha = 2(\sigma_1^2 + \sigma'^2)/\sigma_1(6\sigma_1^2 + \sigma'^2)^{\frac{1}{2}}. \tag{5.6}$$

We now write $$\sigma'/\sigma_1 = \xi, \tag{5.7}$$

so that

$$\left.\begin{aligned} \sigma_2/\sigma_1 &= 1+\xi, & (2\sigma_1-\sigma_2)/\sigma_1 &= 1-\xi, \\ k_2/k_1 &= \sigma_2^2/\sigma_1^2 = (1+\xi)^2, & |2\mathbf{k}_1-\mathbf{k}_2|/k_1 &= (1-\xi)^2, \\ k'/k_1 &= gk'/\sigma_1^2 = \xi(6+\xi^2)^{\frac{1}{2}}, & \cos\alpha &= 2(1+\xi^2)/(6+\xi^2)^{\frac{1}{2}}. \end{aligned}\right\} \tag{5.8}$$

Since k'/k_1 is non-negative, we must take the positive or negative sign in the square root according as ξ is positive or negative. We have

$$\left.\begin{aligned} (k'/k_1)\cos\alpha &= 2\xi(1+\xi^2) = X, \\ (k'/k_1)\sin\alpha_1^2 &= \pm\,\xi[(2+\xi^2)(1+4\xi^2)]^{\frac{1}{2}} = Y, \end{aligned}\right\} \tag{5.9}$$

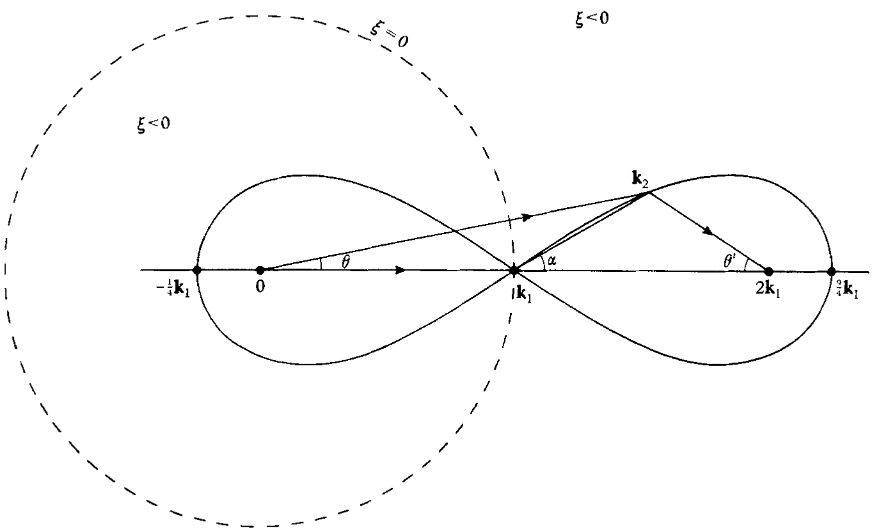

FIGURE 2. Locus of $\mathbf{k}_2$, when $\mathbf{k}_1$ is fixed.

and a plot of Y *vs.* X readily yields the figure-of-eight diagram obtained by Phillips (1960), and shown in figure 2. Positive values of ξ correspond to points on the right-hand loop, and negative values of ξ to points on the left-hand loop. The gradient of the curve at the centre point ($\xi = 0$) is

$$\pm\,1/\sqrt{2} = \pm\tan 35^\circ\,16'. \tag{5.10}$$

The two end-points correspond to $\xi = \pm\frac{1}{2}$, i.e.

$$\sigma_2/\sigma_1 = \tfrac{3}{2} \text{ or } \tfrac{1}{2}, \quad k_2/k_1 = \tfrac{9}{4} \text{ or } \tfrac{1}{4}. \tag{5.11}$$

The angle θ between $\mathbf{k}_1$ and $\mathbf{k}_2$ is found from figure 2 by noting that

$$k_2\cos\theta = k_1 + k'\cos\alpha, \tag{5.12}$$

and so

$$\cos\theta = \frac{k_1}{k_2}(1+X) = \frac{1+2\xi+2\xi^3}{(1+\xi)^2}. \tag{5.13}$$

Similarly, the angle θ' between $(2\mathbf{k}_1-\mathbf{k}_2)$ and $\mathbf{k}_1$ is given by

$$|2\mathbf{k}_1-\mathbf{k}_2|\cos\theta' = k_1 - k'\cos\alpha \tag{5.14}$$

and hence

$$\cos\theta' = \frac{k_1}{|2\mathbf{k}_1-\mathbf{k}_2|}(1-X) = \frac{1-2\xi-2\xi^3}{(1-\xi)^2}. \tag{5.15}$$

A plot of $\pm\,\theta$ and $\pm\,\theta'$ is shown in figure 3, for $\xi \geqslant 0$. If the sign of ξ is reversed, the two curves are simply interchanged.

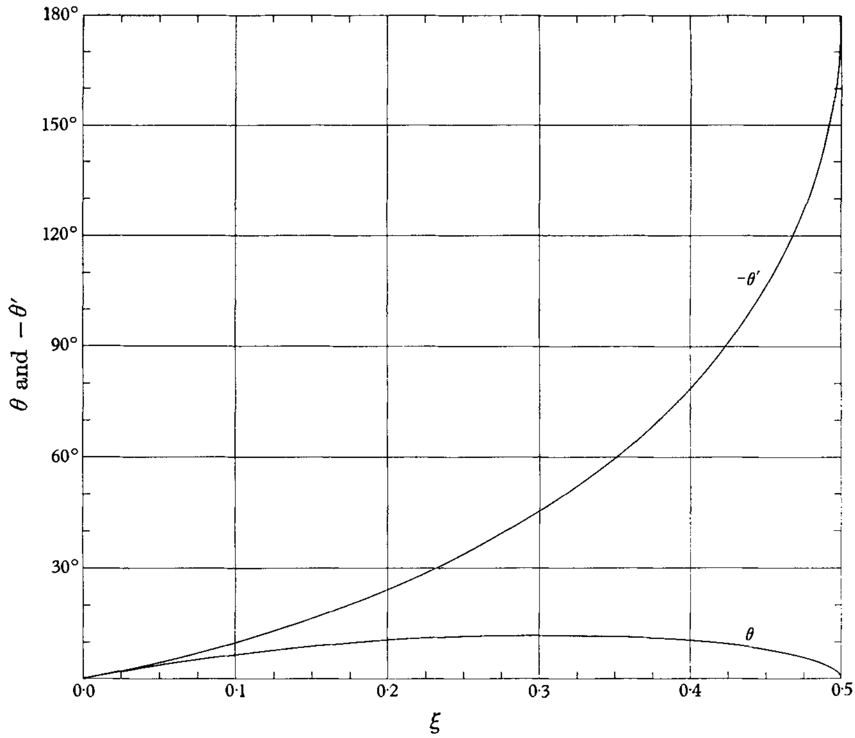

FIGURE 3. Corresponding values of θ and $-\theta'$ as functions of ξ, when $\xi \geqslant 0$.

6. Evaluation of K

From equation (5.13) we have

$$\cos^2 \tfrac{1}{2}\theta = \frac{(1+2\xi)(2+\xi^2)}{2(1+\xi)^2}, \tag{6.1}$$

and from (5.8)

$$\cos^2 \tfrac{1}{2}\alpha = \tfrac{1}{2}\{1+2(1+\xi^2)(6+\xi^2)^{-\frac{1}{2}}\}. \tag{6.2}$$

On substituting these values, and others derived from (5.8), into equation (4.2) we find, after some reduction,

$$K = (a_1 k_1)^2 (a_2 k_2) g^2 \sigma_1^{-1} F(\xi), \tag{6.3}$$

where

$$F(\xi) \equiv \frac{(1+\frac{1}{2}\xi^2)^2(1-4\xi^2)}{(1+\xi)^3}\left[1+\frac{4\xi}{\xi-(6+\xi^2)^{\frac{1}{2}}}\right]. \tag{6.4}$$

The amplitude of the tertiary wave is given by

$$|\zeta_{21}| = \tfrac{1}{2}Ktg^{-1} = (a_1 k_1)^2 (a_2 k_2) \tfrac{1}{2} gt\sigma_1^{-1} F(\xi), \tag{6.5}$$

so that its ratio to the amplitude a_1 of the first wave is

$$|\zeta_{21}|/a_1 = \tfrac{1}{2}(a_1 k_1)(a_2 k_2)(\sigma_1 t) F(\xi). \tag{6.6}$$

When the two primary wave-numbers are equal ($\mathbf{k}_2 = \mathbf{k}_1$), we have $\xi = 0$ and $F = 1$, so that

$$\zeta_{21} = \tfrac{1}{2}a_1^2 a_2 k_1^2 \sigma_1 t \sin(k_1 x - \sigma_1 t). \tag{6.7}$$

In this case only ζ_{21} can be interpreted as a contribution to the increase in frequency of the primary wave $(\zeta_{10}+\zeta_{01})$ (see Phillips 1960). In the case $\xi = \frac{1}{2}$, when $\mathbf{k}_2$ lies at the right-hand extremity of the figure-of-eight, we see from (6.4) that $F = 0$. That is to say, in the only other case when the waves $\mathbf{k}_1$ and $\mathbf{k}_2$ are propagated in the same direction, the amplitude of the tertiary wave vanishes. Similarly, when $\xi = -\frac{1}{2}$, that is to say when $\mathbf{k}_1$ lies at the left-hand end of the figure-of-eight and the directions of $\mathbf{k}_1$ and $\mathbf{k}_2$ are opposite, then F also vanishes.

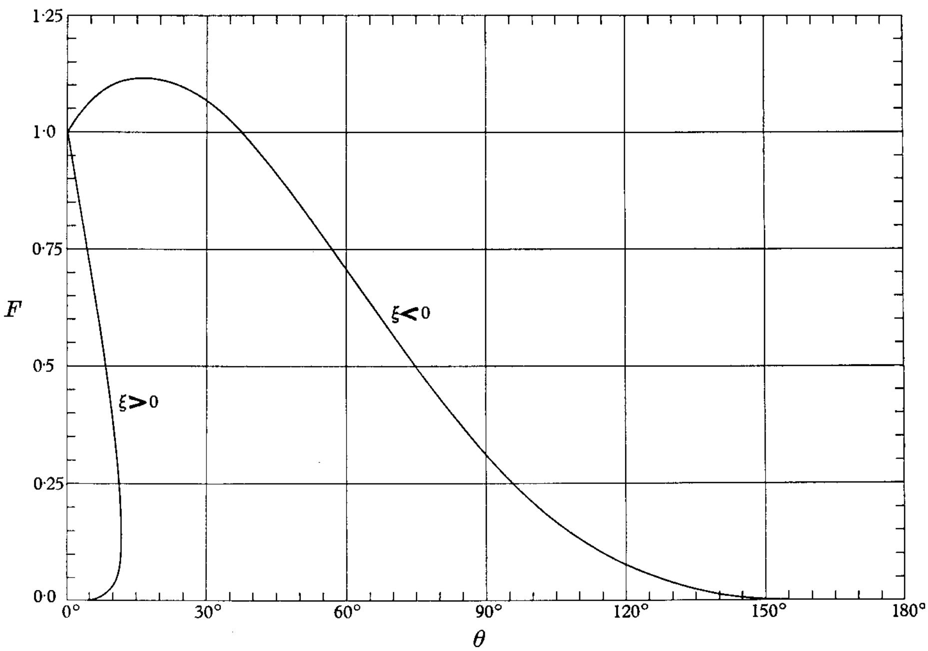

FIGURE 4. The coupling function F, giving the time rate of growth of the tertiary wave, under spacially uniform conditions.

A plot of F *vs.* θ is shown in figure 4. It will be seen that for $\xi < 0$, i.e. on the left-hand part of the figure-of-eight, the coupling is much greater than for $\xi > 0$. A maximum of F occurs at around $\theta = 17°$ on the left-hand part of the loop.

When the two primary waves are at right-angles, $\cos\theta$ vanishes and so ξ is the real root of

$$2\xi^3 + 2\xi + 1 = 0, \tag{6.8}$$

that is to say

$$\xi = \{\tfrac{1}{4}(\sqrt{\tfrac{43}{27}} - 1)\}^{\frac{1}{3}} - \{\tfrac{1}{4}(\sqrt{\tfrac{43}{27}} + 1)\}^{\frac{1}{3}} \doteqdot -0{\cdot}42385, \tag{6.9}$$

and the ratio of σ_1 to σ_2 is $1 : 0{\cdot}57615$, or $1{\cdot}7357 : 1$. From figure 4 we have then

$$F(\xi) \doteqdot 0{\cdot}312. \tag{6.10}$$

7. Finite fetch

Suppose that the two wave trains interact for an unlimited time, but over only a finite distance D, measured in the direction of propagation of the tertiary wave. Then it is necessary only to replace t in equation (6.5) by D/c_g, where c_g is the group-velocity of the tertiary wave:

$$c_g = \frac{g}{2(2\sigma_1 - \sigma_2)} = \frac{g}{2\sigma_1(1-\xi)}. \tag{7.1}$$

Thus
$$\zeta_{21} = (a_1 k_1)^2 (a_2 k_2) D . G(\xi), \tag{7.2}$$

where
$$G(\xi) \equiv (1-\xi) F(\xi). \tag{7.3}$$

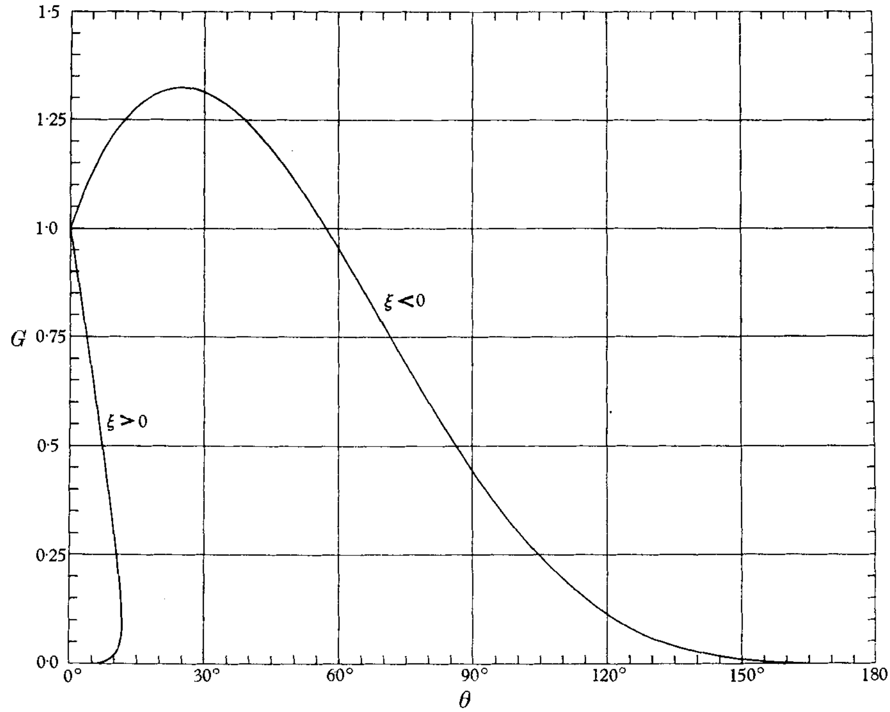

FIGURE 5. The coupling function G, giving the space rate of growth of the tertiary wave, under steady conditions.

It is remarkable that $|\zeta_{21}|$, as given by (7.2) depends only on the maximum slopes of the two primary wave trains, the total distance D and the angle of intersection (given by the parameter ξ).

The function G is shown plotted against θ in figure 5. Like F, it is greater over the left-hand part of the figure-of-eight ($\xi < 0$), and has a maximum ($G \doteqdot 1{\cdot}32$) at around $\theta = 24°$. When $\theta = 90°$ (the two primary waves are at right-angles), we see from figure 5 that

$$G \doteqdot 0{\cdot}442. \tag{7.4}$$

8. A proposed experiment

In a rectangular basin, suppose that wave-makers are placed along two adjacent sides AB and DA (see figure 6), and that beaches, or other efficient wave absorbers, are placed along the other two sides. On the side AB let a wave train of frequency σ_1 be generated, travelling directly across the tank, and likewise on the side DA a wave of frequency σ_2.

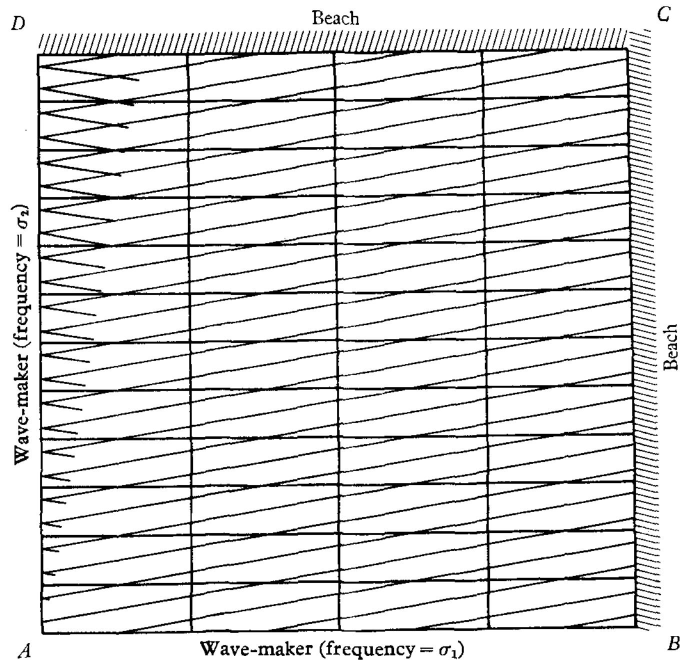

FIGURE 6. The pattern of wave crests in the proposed experiment.

Generally the two wave trains will excite no resonant response; but when the frequencies are in the critical ratio 1·736 : 1 a tertiary wave should be generated travelling in a direction making 9° 24′ with the direction of $\mathbf{k}_1$, and slightly towards the second wave-maker (see figure 6); its amplitude should increase linearly in the direction of propagation. The tertiary wave will be reflected from the wave-maker at the side of the tank, but the reflected wave will be appreciable only in a narrow wedge-shaped zone as on the left of figure 6.†

There may of course be other non-linear effects (for example at the beaches) which will give rise to frequencies $(2\sigma_1 - \sigma_2)$. However, risk of confusion with any such effects may be eliminated by the following procedure. Let the frequency of one primary wave (say σ_1) be kept fixed while the frequency of the other is slightly changed. One may expect that the amplitude of the tertiary wave will be proportional to

$$|\sin(D\,.\,\delta k)/D\,.\,\delta k|, \tag{8.1}$$

where $\delta k = \frac{1}{2}[|2\mathbf{k}_1 - \mathbf{k}_2| - (2\sigma_1 - \sigma_2)^2]$ and D is the distance over which the interaction occurs.

† This reflexion could be eliminated by inclining the second wave-maker at an angle of about 80° to the first, instead of at 90°.

Although the tertiary wave may not be visible to the eye, it should be possible to detect it by a harmonic analysis of a wave record at any fixed point over a sufficient length of time. For example, if D is 10 ft. and the maximum steepness in each wave train is 1/10, then equation (7.2) gives

$$|\zeta_{21}| = 0{\cdot}442 \times 10 \times 10^{-3}\,\text{ft.} = 0{\cdot}05\,\text{in.},$$

which should be readily detectable.

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